

# Quantum Observable Generalized Orthoalgebras\*

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**Abstract.** *Given a Hilbert space  $\mathcal{H}$ , we introduce a binary relation  $\perp$  between two self-adjoint operators  $A$  and  $B$ . We show that if  $A \perp B$ , then  $A$  and  $B$  are affiliated with some abelian von Neumann algebra. We prove that the relation  $\perp$  induces a partial algebraic operation  $\oplus$  on the set  $\mathcal{S}(\mathcal{H})$  of all the self-adjoint operators on  $\mathcal{H}$ , and  $(\mathcal{S}(\mathcal{H}), \perp, \oplus, 0)$  is a generalized orthoalgebra. This conclusion show that the set of all physical quantities on  $\mathcal{H}$  have important mathematics structures, in particular,  $(\mathcal{S}(\mathcal{H}), \perp, \oplus, 0)$  has a partially order  $\preceq$ , we prove that  $A \preceq B$  if and only if  $A$  has a value in  $\Delta$  implies that  $B$  has a value in  $\Delta$  for each real Borel set  $\Delta$  not containing 0. Moreover, we study the existence of the infimum  $A \wedge B$  and the supremum  $A \vee B$  for  $A, B \in \mathcal{S}(\mathcal{H})$  with respect to  $\preceq$ . We show that the famous position operator  $Q$  and momentum operator  $P$  satisfy that  $Q \wedge P = 0$ .*

**Key Words.** *Quantum observable, Affiliated, Generalized orthoalgebra, Order.*

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# 1 Introduction

In 1900, Hilbert presented 23 famous problems. The sixth problem is: To find a few physical axioms which, similar to the axioms of geometry, can describe a theory for a class of physical events that is as large as possible. In 1933, Kolmogorov axiomatized modern probability theory ([1]). In Kolmogorov's theory, the set  $\mathcal{L}$  of experimentally verifiable events forms a Boolean  $\sigma$ -algebra. Thus, Boolean algebra theory can describe classical logic. However, Kolmogorov's theory does not describe the situation that arises from quantum mechanics, e.g. the famous Heisenberg uncertainty principle ([2]). One of the most important problems in quantum theory is to find a mathematical description of the structure of random events of a quantum system. The problem was originally studied in 1936 by Birkhoff and von Neumann ([3]). The main difficulty is that many quantum phenomenon cannot be described in terms of the event structure in classical probability. In von Neumann's approach, a quantum system is represented by a separable complex Hilbert space  $\mathcal{H}$ , each physical quantity is represented by a self-adjoint operator on  $\mathcal{H}$ , and called it a quantum observable, the set of all the quantum observables is denoted by  $\mathcal{S}(\mathcal{H})$ . Since the spectrum  $\sigma(P)$  of a projection operator  $P$  is contained in  $\{0, 1\}$ , if the truth values, false and true, for two-values propositions about the quantum system are encoded by 0 and 1, then these propositions can correspond to projection operators on  $\mathcal{H}$ . That is, two-valued propositions about the quantum system can be represented by projection operators on  $\mathcal{H}$ . Birkhoff and von Neumann considered the set  $\mathcal{L}(\mathcal{H})$  of all projection operators on  $\mathcal{H}$  as the logic of the quantum system ([3]). For each  $A \in \mathcal{S}(\mathcal{H})$ , if  $P^A$  is the spectrum measure of  $A$ , then for each real Borel set  $\Delta$ ,  $P^A(\Delta)$  represents the event that the values of physical quantity  $A$  are contained in  $\Delta$ .

Let  $\mathcal{L}$  be a lattice with two binary operations the supremum  $\vee$  and the infimum  $\wedge$ . If there are two elements 0 and  $I$  in  $\mathcal{L}$  and a unary operation  $' : \mathcal{L} \rightarrow \mathcal{L}$  such that  $x'' = x, x \vee x' = I, x \wedge x' = 0$  for all  $x \in \mathcal{L}$  such that  $0 \leq x \leq I$ , then  $(\mathcal{L}, \vee, \wedge, ', 0, I)$  is said to be an ortholattice, and  $'$  is said to be an orthocomplementation operation.

We say the ortholattice  $(\mathcal{L}, \vee, \wedge, ', 0, I)$  satisfies the orthomodular law if

$$x \leq y \Rightarrow y = x \vee (y \wedge x'),$$

whenever  $x, y \in \mathcal{L}$ .

The name of orthomodular was suggested by Kaplansky. An ortholattice satisfying the orthomodular law is said to be an orthomodular lattice ([4]).

We say the ortholattice  $(\mathcal{L}, \vee, \wedge, ', 0, I)$  satisfies the modular law if

$$x \leq y \Rightarrow y \wedge (x \vee z) = x \vee (z \wedge y),$$

whenever  $x, y, z \in \mathcal{L}$ . An ortholattice, which satisfies the modular law, is said to be a modular lattice ([4]).

Let  $p, q \in \mathcal{L}(\mathcal{H})$ . We say  $p \leq q$  if  $\langle px, x \rangle \leq \langle qx, x \rangle$  for all  $x \in \mathcal{H}$ .  $(\mathcal{L}(\mathcal{H}), \leq)$  is a lattice with respect to the partial order  $\leq$  and has the minimal element 0 and the maximal element  $I$ . Moreover, if we define  $p' = I - p$ , then  $(\mathcal{L}(\mathcal{H}), \vee, \wedge, ', 0, I)$  is an ortholattice.

In 1937, Husimi showed that  $(\mathcal{L}(\mathcal{H}), \vee, \wedge, ', 0, I)$  has the orthomodular law ([5]), thus,  $(\mathcal{L}(\mathcal{H}), \vee, \wedge, ', 0, I)$  is an orthomodular lattice.

People can also consider the orthomodular lattices as the quantum logics, see [4].

Moreover, Birkhoff and von Neumann showed that if  $\mathcal{H}$  is a finite dimensional space, then  $(\mathcal{L}(\mathcal{H}), \vee, \wedge, ', 0, I)$  is a modular lattice ([3]).

Note that each Boolean algebra  $\mathcal{A}$  is a distributive ortholattice. That is, for  $x, y, z \in \mathcal{A}$ , we have

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z),$$

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z).$$

Therefore, we see that the quantum logic  $(\mathcal{L}(\mathcal{H}), \vee, \wedge, ', 0, I)$  is completely different from the classical logic.

Let  $(\mathcal{L}, \vee, \wedge, ', 0, I)$  be an orthomodular lattice. We say that  $x$  and  $y$  satisfy the binary relation  $\perp$  if  $x \leq y'$ . We define a partial operation  $\oplus$  on  $\mathcal{L}$  by  $x \oplus y = x \vee y$

if  $x \perp y$ . Then, we get a new algebraic structure  $(\mathcal{L}, \perp, \oplus, 0, I)$  with the following properties:

- (OA1) If  $x \perp y$ , then  $y \perp x$  and  $x \oplus y = y \oplus x$ .
- (OA2) If  $y \perp z$  and  $x \perp (y \oplus z)$ , then  $x \perp y$ ,  $(x \oplus y) \perp z$  and  $(x \oplus y) \oplus z = x \oplus (y \oplus z)$ .
- (OA3) For each  $x \in \mathcal{L}$ , there exists a unique  $y \in \mathcal{L}$  such that  $x \perp y$  and  $x \oplus y = I$ .
- (OA4) If  $x \perp x$ , then  $x = 0$ .

In 1992, Foulis, Greechie and Ruttimann called the algebraic structure  $(\mathcal{L}, \perp, \oplus, 0, I)$  an orthoalgebra ([6]).

In 1996, Kalmbach, Riečanová, Hedlíková and Pulmannová, Dvurečenskij introduced the following equivalent definition ([7, 8, 9]):

**Definition 1.** A generalized orthoalgebra  $(\mathcal{E}, \perp, \oplus, 0)$  is a set  $\mathcal{E}$  with an element 0, a binary relation  $\perp$ , and a partial operation  $\oplus$ , such that if  $x \perp y$ , then  $x \oplus y$  is defined and satisfies the following conditions:

- (OA1). If  $x \perp y$ , then  $y \perp x$  and  $x \oplus y = y \oplus x$ .
- (OA2). If  $y \perp z$  and  $x \perp (y \oplus z)$ , then  $x \perp y$ ,  $(x \oplus y) \perp z$  and  $(x \oplus y) \oplus z = x \oplus (y \oplus z)$ .
- (GOA3). If  $x \perp y$ ,  $x \perp z$  and  $x \oplus y = x \oplus z$ , then  $y = z$ .
- (GOA4). If  $x \perp y$  and  $x \oplus y = 0$ , then  $x = y = 0$ .
- (GOA5).  $x \perp 0$  and  $x \oplus 0 = x$  for all  $x \in \mathcal{E}$ .
- (GOA6). If  $x \perp x$ , then  $x = 0$ .

Let  $a, b \in \mathcal{E}$ . If there is a  $c \in \mathcal{E}$  such that  $a \perp c$  and  $a \oplus c = b$ , then we say that  $a \preceq b$ . It can be proven that  $\preceq$  is a partial order, namely, the generalized orthoalgebra  $(\mathcal{E}, \perp, \oplus, 0)$  has a natural partial order  $\preceq$ . Moreover,  $x \perp y$  if and only if  $x \leq y'$  ([9]).

Generalized orthoalgebras are very important models of quantum logic. They extend the quantum logic  $(\mathcal{L}(\mathcal{H}), \vee, \wedge, ', 0, I)$  greatly ([9]).

In [10], Gudder defined a binary relation  $\perp$  on the set  $S_b(\mathcal{H})$  of all bounded self-adjoint operators on  $\mathcal{H}$  by  $A \perp B$  once  $AB = 0$ , in which case, define  $A \oplus B = A + B$ . However, it is unfortunate for the world we live in that many of the operators that arise naturally are not bounded. For example, in the famous Heisenberg's commutation relation  $QP - PQ = -i\hbar I$ , the position operator  $Q$

and the momentum operator  $P$  are both unbounded self-adjoint operators ([11]). Therefore, it is necessary to study such operators and the set  $\mathcal{S}(\mathcal{H})$  of all bounded and unbounded selfadjoint operators on  $\mathcal{H}$ .

In this paper, we introduce a binary relation  $\perp$  on  $\mathcal{S}(\mathcal{H})$  such that  $A \perp B$  iff  $\overline{ran}(A)$  is orthogonal to  $\overline{ran}(B)$ , where  $ran(A)$  denotes the range of  $A$ . If  $A \perp B$ , we define  $A \oplus B = A + B$ . We show that if  $A \perp B$ , then  $A$  and  $B$  are affiliated with some abelian von Neumann algebra. Moreover, we show that  $(\mathcal{S}(\mathcal{H}), \perp, \oplus, 0)$  is a generalized orthoalgebra. Thus, we establish a new quantum logic structure on all physics quantities of the quantum system  $\mathcal{H}$ . Note that the generalized orthoalgebra  $(\mathcal{S}(\mathcal{H}), \perp, \oplus, 0)$  has a nature partial order  $\preceq$ , we will show that  $A \preceq B$  iff  $A$  has a value in  $\Delta$  implies that  $B$  has a value in  $\Delta$  for each real Borel set  $\Delta$  not containing 0. Moreover, we study the existence of the infimum  $A \wedge B$  and supremum  $A \vee B$  for  $A, B \in \mathcal{S}(\mathcal{H})$  with respect to  $\preceq$ . In the end, we will show that the position operator  $Q$  and momentum operator  $P$  satisfy that  $Q \wedge P = 0$ .

## 2 Definitions and Facts of Self-adjoint Operators

We first recall some elementary concepts and facts of unbounded linear operators, see [12]. An unbounded linear operator  $A$  we consider will have a domain  $\mathcal{D}(A)$  that is a dense linear subspace of  $\mathcal{H}$ . Given two linear operators  $A : \mathcal{D}(A) \rightarrow \mathcal{H}$  and  $B : \mathcal{D}(B) \rightarrow \mathcal{H}$ , we write  $A \subseteq B$  and say that  $B$  is an extension of  $A$ , if  $\mathcal{D}(A) \subseteq \mathcal{D}(B)$  and  $Ax = Bx$  for all  $x \in \mathcal{D}(A)$ . To each linear operator  $A$ , there exists a linear operator  $A^*$  of  $A$  such that  $\mathcal{D}(A^*) = \{y \in \mathcal{H} : \text{there exists } y^* \in \mathcal{H} \text{ such that } \langle y^*, x \rangle = \langle y, Ax \rangle \text{ for all } x \in \mathcal{D}(A)\}$ , and define  $A^*y = y^*$  for each  $y \in \mathcal{D}(A^*)$ . If  $A \subseteq A^*$ , then  $A$  is said to be a symmetric operator. If  $A = A^*$ , then  $A$  is said to be a self-adjoint operator.

For each  $A$ , we associate  $A$  with a graph  $G(A) = \{(x, Tx) | x \in \mathcal{D}(A)\}$ . We say  $A$  is closed if  $G(A)$  is a closed set, and  $A$  is closable if there exists a closed linear operator  $B$  such that  $A \subseteq B$  and  $\overline{G(A)} = G(B)$ . It is well known that the domain  $\mathcal{D}(A) \neq \mathcal{H}$  for each unbounded closed linear operator  $A$ .

Each self-adjoint operator is closed and each symmetric operator is closable.

Suppose that  $A$  is closable. Let  $\overline{A}$  denote the closed linear operator such that  $G(\overline{A}) = \overline{G(A)}$ . The operator  $\overline{A}$  is called the closure of  $A$ .

From the viewpoint of calculations with an unbounded operator  $A$ , it is often much easier to study its restriction  $A|_{\mathcal{D}_0}$  to a dense linear subspace  $\mathcal{D}_0$  in its domain  $\mathcal{D}(A)$  than to study  $A$  itself. If  $A$  is closed and  $\overline{G(A|_{\mathcal{D}_0})} = G(A)$ , the information obtained in this way is much more applicable to  $A$ . In this case, we say that  $\mathcal{D}_0$  is a core for  $A$ .

A family  $\{E_\lambda\}$  of projection operators indexed by  $\mathbf{R}$ , satisfying

(i).  $\bigwedge_{\lambda \in \mathbf{R}} E_\lambda = 0$  and  $\bigvee_{\lambda \in \mathbf{R}} E_\lambda = I$ ,

(ii).  $E_{\lambda_1} \leq E_{\lambda_2}$  if  $\lambda_1 \leq \lambda_2$ ,

(iii).  $\bigwedge_{\lambda \geq \lambda_1} E_\lambda = E_{\lambda_1}$ ,

is said to be a resolution of the identity.

The following is a spectral theory for self-adjoint operators.

**Lemma 1** ([12], [13]). If  $A$  is a self-adjoint operator on  $\mathcal{H}$ , then there is a unique spectral measure  $P^A$  defined on the all Borel subsets of  $\mathbf{R}$  such that

$$A = \int_{\mathbf{R}} \lambda dP^A(\lambda).$$

If we denote by  $E_\lambda^A = P^A((-\infty, \lambda])$ , then  $\{E_\lambda^A\}$  is a resolution of the identity, and it is said to be the resolution of the identity for  $A$ . Moreover, if we denote  $F_n^A = E_n^A - E_{-n}^A$ , then  $\bigcup_{n=1}^\infty F_n^A(\mathcal{H})$  is a core for  $A$ , and

$$Ax = \int_{-n}^n \lambda dE_\lambda^A x$$

for each  $x$  in  $F_n^A(\mathcal{H})$  and all  $n$ , in the sense of norm convergence of approximating Riemann sums.

**Lemma 2** ([12]). If  $A : \mathcal{H} \rightarrow \mathcal{H}$  is a closed operator, then the null space  $null(A) = \{x \in \mathcal{D}(A) : Ax = 0\}$  of  $A$  is a closed subspace of  $\mathcal{H}$ . Moreover,  $(ran(A))^\perp = null(A^*)$ ,  $(ran(A^*))^\perp = null(A)$ ,  $\overline{ran}(A^*A) = \overline{ran}(A^*)$ ,  $null(A^*A) = null(A)$ .

Let  $A \in S(\mathcal{H})$ . It is easy to see that  $F_n^A F_m^A = F_m^A F_n^A$ ,  $F_n^A A \subseteq A F_n^A$ , and  $A F_n^A x \rightarrow Ax$  for each  $x \in \mathcal{D}(A)$  and  $m, n \in \mathbf{N}$ .

Now, we denote  $P_A$  and  $N_A$  to be the projection operators whose ranges are  $\overline{\text{ran}}(A)$  and  $\text{null}(A)$ , respectively.

**Lemma 3** ([12]). Let  $A, B \in S(\mathcal{H})$ . Then  $P^A(\{0\}) = N_A$ ,  $P^A(\mathbf{R} \setminus \{0\}) = P_A$ ,  $P_A + N_A = I$  and  $P_A \vee P_B = I - N_A \wedge N_B$ .

**Lemma 4** ([14]). Let  $A$  be a self-adjoint operator on  $\mathcal{H}$ . If  $S$  is a bounded linear operator on  $\mathcal{H}$  and  $SA \subseteq AS$ ,  $\Delta \subseteq \mathbf{R}$  is a Borel set, then  $P^A(\Delta)S = SP^A(\Delta)$ .

**Lemma 5.** Let  $A, B \in S(\mathcal{H})$ . Then, the following statements are equivalent.

- (i)  $A \perp B$ , that is,  $\overline{\text{ran}}(A)$  is orthogonal to  $\overline{\text{ran}}(B)$ .
- (ii)  $\overline{\text{ran}}(A) \subseteq \text{null}(B)$ .
- (iii)  $\overline{\text{ran}}(B) \subseteq \text{null}(A)$ .
- (iv)  $AB \subseteq 0$  and  $\mathcal{D}(AB) = \mathcal{D}(B)$ .
- (v)  $BA \subseteq 0$  and  $\mathcal{D}(BA) = \mathcal{D}(A)$ .

**Proof** (i)  $\Leftrightarrow$  (ii)  $\Leftrightarrow$  (iii) are trivial.

(ii)  $\Leftrightarrow$  (v): Suppose  $BA \subseteq 0$  and  $\mathcal{D}(BA) = \mathcal{D}(A)$ . For each  $x \in \mathcal{D}(A)$ ,  $Ax \in \mathcal{D}(B)$  and  $BAx = 0$ . That is  $Ax \in \text{null}(B)$ . So  $\text{ran}(A) \subseteq \text{null}(B)$ . Since  $\text{null}(B)$  is closed,  $\overline{\text{ran}}(A) \subseteq \text{null}(B)$ . Conversely, suppose that  $\overline{\text{ran}}(A) \subseteq \text{null}(B)$ . Then, for each  $x \in \mathcal{D}(A)$ ,  $Ax \in \text{null}(B)$ , we have  $x \in \mathcal{D}(BA)$  and  $BAx = 0$ . Therefore,  $\mathcal{D}(BA) = \mathcal{D}(A)$  and  $BA \subseteq 0$ .

Similarly, (iii)  $\Leftrightarrow$  (iv).

### 3 The Affiliate Relationship

We say that a closed densely defined operator  $T$  is affiliated with a von Neumann algebra  $\mathcal{R}$  and write  $T \eta \mathcal{R}$  when  $U^*TU = T$  for each unitary operator  $U$  commuting with  $\mathcal{R}$ . As far as the domains are concerned, the effect is that  $U$  transforms  $\mathcal{D}(T)$  onto itself ([12]).

The affiliation relationship is very important and provides a framework for the formal computation which the physicists made with the unbounded operators.

**Lemma 6** ([12]). If  $A$  is a self-adjoint operator, and  $A$  is affiliated with some abelian von Neuman algebra  $\mathcal{R}$ , then  $\{E_\lambda^A\} \subseteq \mathcal{R}$ .

**Lemma 7** ([12]). If  $\{E_\lambda\}$  is a resolution of the identity,  $\mathcal{R}$  is an abelian von Neumann algebra containing  $\{E_\lambda\}$ , then there is a self-adjoint operator  $A$  is affiliated

with  $\mathcal{R}$ , and

$$Ax = \int_{-n}^n \lambda dE_\lambda x$$

for each  $x$  in  $F_n(\mathcal{H})$  and all  $n$ , where  $F_n = E_n - E_{-n}$ ; and  $\{E_\lambda\}$  is the resolution of the identity for  $A$ .

**Example** ([12]). If  $(S, \varphi, m)$  is a  $\sigma$ -finite measure space and  $\mathcal{A}$  is its multiplication algebra acting on  $L^2(S, m)$ , then  $A$  is a closed operator, and  $A$  affiliated with  $\mathcal{A}$  iff  $A = M_g$  for some measurable function  $g$  finite almost everywhere on  $S$ . In this case,  $A$  is self-adjoint if and only if  $g$  is real-valued almost everywhere.

In 2012, Kadison pointed out the Heisenberg relation  $QP - PQ = -i\hbar I$  cannot be satisfied in the algebra of operators affiliated with any finite von Neumann algebra ([11]).

In this section, our main result is:

**Theorem 1.** If  $A, B \in S(\mathcal{H})$ ,  $A \perp B$ , then there exists an abelian von Neumann algebra  $\mathcal{R}$  such that  $A\eta\mathcal{R}$  and  $B\eta\mathcal{R}$ . Moreover,  $\bigcup_{n=1}^\infty F_n^A F_n^B(\mathcal{H})$  is the common core for  $A$  and  $B$ .

**Proof.**  $A \perp B$  implies that  $\overline{\text{ran}}(A) \perp \overline{\text{ran}}(B)$ . It follows that  $AF_n^A BF_m^B = BF_m^B AF_n^A = 0$  for each  $m, n \in \mathbf{N}$ . For each  $x \in \mathcal{D}(B)$ , as  $BF_m^B x \rightarrow Bx$ , we have  $AF_n^A B \subseteq BAF_n^A$ . From Lemma 4,  $AF_n^A F_m^B = F_m^B AF_n^A$  for each  $m, n \in \mathbf{N}$ . Similarly, we have  $BF_m^B F_n^A = F_n^A BF_m^B$ . Also, it is easy to check that  $F_n^A F_m^B = F_m^B F_n^A$ . Moreover,  $\bigcup_{n=1}^\infty F_n^A F_n^B(\mathcal{H})$  is the common core for  $A$  and  $B$ .

Denote  $\mathcal{R}$  the von Neumann algebra generated by the set  $\{F_n^A, AF_n^A, F_n^B, BF_n^B : n = 1, 2, \dots\}$ . Since the elements in  $\{F_n^A, AF_n^A, F_n^B, BF_n^B : n = 1, 2, \dots\}$  are commuting,  $\mathcal{R}$  is abelian. If  $U$  is a unitary operator in  $\mathcal{R}'$  and  $x \in \bigcup_{n=1}^\infty F_n^A(\mathcal{H})$ ,  $AUx = AU F_n^A x = AF_n^A Ux = UAF_n^A x = UAx$  for some  $n$ . So  $A\eta\mathcal{R}$ . It is the same that  $B\eta\mathcal{R}$ .

## 4 Generalized Orthoalgebra $(\mathcal{S}(\mathcal{H}), \perp, \oplus, 0)$

Now, we prove that  $(\mathcal{S}(\mathcal{H}), \perp, \oplus, 0)$  is a generalized orthoalgebra.

**Proposition 1.** Let  $A, B \in \mathcal{S}(\mathcal{H})$  with  $A^2 = BA$ . Then

- (1).  $\bigcup_{n=1}^{\infty} F_n^B(\mathcal{H})$  is the common core for  $A$  and  $B$ .
- (2).  $\mathcal{D}(B) \subseteq \mathcal{D}(A)$ .

**Proof.** (1). As  $A^2$  is self-adjoint and  $A^2 = BA$ ,  $BA$  is self-adjoint and  $BA \supseteq AB$ .  $A^2 = BA$  implies that  $(AF_n^A)^2 = BAF_n^A \supseteq AF_n^AB$  for each  $n \in \mathbf{N}$ . From Lemma 4,  $F_n^BAF_n^A = AF_n^AF_n^B$ . For each  $x \in \mathcal{D}(A)$  and  $n \in \mathbf{N}$ , since  $AF_m^Ax \rightarrow Ax$  as  $m \rightarrow \infty$ ,  $F_n^BAx = F_n^B(\lim_m AF_m^Ax) = \lim_m F_n^BAF_m^Ax = \lim_m AF_m^AF_n^Bx$ . As  $A$  is closed and  $F_m^AF_n^Bx \rightarrow F_n^Bx$  when  $m \rightarrow \infty$ , we have  $F_n^Bx \in \mathcal{D}(A)$  and  $AF_n^Bx = F_n^BAx$ . That is  $F_n^BA \subseteq AF_n^B$  for each  $n \in \mathbf{N}$ . So  $\bigcup_{n=1}^{\infty} F_n^B(\mathcal{H})$  is the core for  $A$ . It follows that  $\bigcup_{n=1}^{\infty} F_n^B(\mathcal{H})$  is the common core for  $A$  and  $B$ .

(2). Since  $F_n^BA \subseteq AF_n^B$  and  $F_n^BA$  is closable for each  $n \in \mathbf{N}$ , it is easy to prove that  $\overline{F_n^BA} = AF_n^B$ . Thus  $(F_n^BA)^* = (\overline{F_n^BA})^* = (AF_n^B)^*$  and it follows that  $F_n^BA \subseteq (AF_n^B)^* = (F_n^BA)^* = AF_n^B$ . So  $AF_n^B$  is self-adjoint for each  $n \in \mathbf{N}$ . Since  $A^2 = BA$  and  $(AF_n^BF_m^B)^2 = BF_n^BAF_m^B = AF_m^BBF_n^B$  for each  $m, n \in \mathbf{N}$ , we have  $BF_n^BA \subseteq ABF_n^B$ . By Lemma 4,  $BF_n^BP_A = P_ABF_n^B$ . For each  $x \in \mathcal{D}(B)$ ,  $BF_n^Bx \rightarrow Bx$ ,  $BF_n^BP_Ax = P_ABF_n^Bx \rightarrow P_ABx$ . Since  $F_n^BP_Ax \rightarrow P_Ax$  and  $B$  is closed, we have  $P_Ax \in \mathcal{D}(B)$  and  $BP_Ax = P_ABx$ . That is  $P_AB \subseteq BP_A$ .

As  $A^2 = BA \supseteq AB$ , for each  $x \in \mathcal{D}(B)$  and  $Bx = 0$ , it follows that  $x \in \mathcal{D}(A^2)$  and  $A^2x = 0$ . Since  $\text{null}(A^2) = \text{null}(A)$ , we have  $Ax = 0$  and  $\text{null}(B) \subseteq \text{null}(A)$ . Then

$$\mathcal{H} = \overline{\text{ran}}(A) \oplus (\text{null}(A) \cap \overline{\text{ran}}(B)) \oplus \text{null}(B).$$

For each  $x \in \mathcal{D}(B)$ ,  $x = x_1 + x_2 + x_3$  where  $x_1 \in \overline{\text{ran}}(A)$ ,  $x_2 \in (\text{null}(A) \cap \overline{\text{ran}}(B))$  and  $x_3 \in \text{null}(B)$ .  $P_AB(x_1 + x_2) = BP_A(x_1 + x_2) = BP_Ax_1 = Bx_1$ . That is  $x_1 \in \mathcal{D}(B)$ . As  $x_1 \in \overline{\text{ran}}(A)$ , there exists a sequence  $\{x_m\} \subseteq \mathcal{D}(A)$  such that  $Ax_m \rightarrow x_1$ . For each  $n \in \mathbf{N}$ ,  $AF_n^Bx_1 = AF_n^B(\lim_m Ax_m) = \lim_m AF_n^BAx_m = \lim_m BAF_n^Bx_m = \lim_m BF_n^BAF_n^Bx_m = BF_n^B(\lim_m F_n^BAx_m) = BF_n^Bx_1 \rightarrow Bx_1$ . Since  $A$  is closed, we have  $x_1 \in \mathcal{D}(A)$  and  $Ax_1 = Bx_1$ . Thus we have  $x_i \in \mathcal{D}(A)$  for  $i = 1, 2, 3$  and

$x \in \mathcal{D}(A)$ . Therefore,  $\mathcal{D}(B) \subseteq \mathcal{D}(A)$ .

In general, the sum of two unbounded self-adjoint operators is not always self-adjoint. However, we have the following result.

**Proposition 2.** Let  $A, B \in S(\mathcal{H})$  with  $A \perp B$ . Then  $\mathcal{D}(A + B)$  is a dense subspace of  $\mathcal{H}$  and  $A + B$  is a self-adjoint operator.

**Proof.** From the proof of Proposition 1, we know that  $\bigcup_{n=1}^{\infty} F_n^A F_n^B(\mathcal{H})$  is a common core for  $A$  and  $B$ . So  $\mathcal{D}(A + B) = \mathcal{D}(A) \cap \mathcal{D}(B)$  is dense.

Now, we turn to prove that  $A + B$  is closed. Let  $\{x_n\} \subseteq \mathcal{D}(A + B)$  with  $x_n \rightarrow x$  and  $(A + B)x_n \rightarrow y$ . Since  $\mathcal{H} = \overline{\text{ran}}(A) \oplus \text{null}(A)$ , we have  $x_n = x_n^{(1)} + x_n^{(2)}$  where  $\{x_n^{(1)}\} \subseteq \overline{\text{ran}}(A)$  and  $\{x_n^{(2)}\} \subseteq \text{null}(A)$ . As  $\overline{\text{ran}}(A) \subseteq \text{null}(B)$ ,  $(A + B)x_n = Ax_n^{(1)} + Bx_n^{(2)} \rightarrow y$ . Since  $N_B(A + B)x_n = N_B Ax_n^{(1)} + N_B Bx_n^{(2)} \rightarrow N_B y$  and  $N_B Ax_n^{(1)} = Ax_n^{(1)}$ ,  $N_B Bx_n^{(2)} = 0$ , we have  $Ax_n^{(1)} \rightarrow N_B y$  and  $Bx_n^{(2)} \rightarrow y - N_B y$ . Since  $A$  is closed and  $Ax_n = Ax_n^{(1)} \rightarrow N_B y$ , it follows that  $x \in \mathcal{D}(A)$  and  $Ax = N_B y$ . Similarly, we have  $x \in \mathcal{D}(B)$  and  $Bx = y - N_B y$ . Therefore,  $x \in \mathcal{D}(A) \cap \mathcal{D}(B)$  and  $(A + B)x = y$ , which implies that  $A + B$  is closed.

Moreover, we can prove that  $\bigcup_{n=1}^{\infty} F_n^A F_n^B(\mathcal{H})$  is also the core for  $(A + B)^*$ . In fact, as  $F_n^A F_n^B(A + B) \subseteq (A + B)F_n^A F_n^B$  and  $F_n^A F_n^B = F_n^B F_n^A$ , it follows that  $(A + B)^* F_n^A F_n^B \supseteq F_n^A F_n^B (A + B)^*$ . For each  $x \in \mathcal{D}((A + B)^*)$ ,  $F_n^A F_n^B x \rightarrow x$  and  $(A + B)^* F_n^A F_n^B x = F_n^A F_n^B (A + B)^* x \rightarrow (A + B)^* x$ . So  $\bigcup_{n=1}^{\infty} F_n^A F_n^B(\mathcal{H})$  is the core for  $(A + B)^*$ .

Since  $A + B = A^* + B^* \subseteq (A + B)^*$  and they have the common core, we have  $A + B = (A + B)^*$  and  $A + B$  is self-adjoint.

Thus, for  $A, B \in S(\mathcal{H})$ , we can define  $A \oplus B = A + B$  when  $A \perp B$ .

**Theorem 2.**  $(S(\mathcal{H}), \perp, \oplus, 0)$  is a generalized orthoalgebra.

**Proof.** For  $A, B \in S(\mathcal{H})$ , the conditions (OA1) and (GOA5) are trivial.

(OA2). Let  $A \perp B$  and  $A \oplus B \perp C$ . We first show that  $B \perp C$ . For each  $x \in \mathcal{D}(C)$ , by the fact  $(A + B) \perp C$ , we have  $(A + B)Cx = 0$  and  $ACx + BCx = 0$ . Then  $\langle ACx, BCx \rangle + \langle BCx, BCx \rangle = 0$ . As  $A \perp B$ ,  $\langle ACx, BCx \rangle = 0$ . So  $\langle BCx, BCx \rangle = 0$  and  $BCx = 0$ . Thus we have  $\text{ran}(C) \subseteq \text{null}(B)$ . As  $\text{null}(B)$  is

closed, we have  $\overline{\text{ran}}(C) \subseteq \text{null}(B)$ . By Lemma 5,  $B \perp C$ . Similarly, we have  $A \perp C$ . Next, we prove  $A \perp (B + C)$ . For each  $x \in \mathcal{D}(B + C) = \mathcal{D}(B) \cap \mathcal{D}(C)$ ,  $A \perp B$  and  $A \perp C$  implies  $Bx \in \text{null}(A)$  and  $Cx \in \text{null}(A)$ . Thus  $(B + C)x \in \text{null}(A)$ . Then  $\overline{\text{ran}}(B + C) \subseteq \text{null}(A)$ . By Lemma 5 again, we obtain  $B + C \perp A$ . That is  $B \oplus C \perp A$ . It is obvious that  $(A \oplus B) \oplus C = A \oplus (B \oplus C)$ .

(GOA3). Let  $A \oplus B = A \oplus C$ . We have proved that  $\bigcup F_n^A F_n^B(\mathcal{H})$ ,  $\bigcup F_n^A F_n^C(\mathcal{H})$  are the common cores for  $A, B$  and  $A, C$  respectively. Obviously,  $F_n^A F_n^B F_n^C \uparrow$  with the strong operator limit  $I$ , the identity operator, and  $\bigcup F_n^A F_n^B F_n^C(\mathcal{H})$  is dense in  $\mathcal{H}$ . It follows that  $\bigcup F_n^A F_n^B F_n^C(\mathcal{H})$  is the common core for  $A, B, C$ . As  $A \oplus B = A \oplus C$ , namely  $\overline{A + B} = \overline{A + C}$ , we have  $A + B|_{\bigcup F_n^A F_n^B F_n^C(\mathcal{H})} = A + C|_{\bigcup F_n^A F_n^B F_n^C(\mathcal{H})}$ . Thus  $B|_{\bigcup F_n^A F_n^B F_n^C(\mathcal{H})} = C|_{\bigcup F_n^A F_n^B F_n^C(\mathcal{H})}$  and  $B = C$  since  $B$  and  $C$  are the same on their common core.

(GOA4). Suppose  $A \perp B$  and  $A \oplus B = 0$ . That is  $A + B = 0$ . Then  $\overline{\text{ran}}(A) \perp \overline{\text{ran}}(B)$ . For any  $x \in \mathcal{D}(A) \cap \mathcal{D}(B)$ ,  $Ax + Bx = 0$ ,  $\langle Ax + Bx, Ax \rangle = 0$ . Since  $\langle Ax, Bx \rangle = 0$ ,  $\langle Ax, Ax \rangle = 0$  and  $Ax = 0$ . Then  $Ax = 0$  for each  $x \in \mathcal{D}(A)$  since  $\mathcal{D}(A) \cap \mathcal{D}(B)$  is dense in  $\mathcal{H}$ . Then  $A = 0$  since  $A$  is self-adjoint. Similarly, We have  $B = 0$ .

(GOA6). Let  $A \perp A$ . Then  $\overline{\text{ran}}(A) \perp \overline{\text{ran}}(A)$ . So for each  $x \in \mathcal{D}(A)$ ,  $\langle Ax, Ax \rangle = 0$  and  $Ax = 0$ . Then  $A \subseteq 0$ , since  $A$  is self-adjoint,  $A = 0$ .

Hence,  $(\mathcal{S}(\mathcal{H}), \perp, \oplus, 0)$  is a generalized orhtoalgebra.

## 5 The order properties of $(\mathcal{S}(\mathcal{H}), \perp, \oplus, 0)$

Now, we study the order properties of  $(\mathcal{S}(\mathcal{H}), \perp, \oplus, 0)$ . That is, for  $A, B \in \mathcal{S}(\mathcal{H})$  we have  $A \preceq B$  if there exists a  $C \in \mathcal{S}(\mathcal{H})$  such that  $A \perp C$  and  $A \oplus C = B$ . It is clear  $0 \preceq A$  for each  $A \in \mathcal{S}(\mathcal{H})$ .

**Proposition 3.** Let  $A, B \in \mathcal{S}(\mathcal{H})$ . Then,  $A \preceq B$  iff  $A^2 = BA$ .

**Proof.** Necessity. Suppose  $A \preceq B$ . There is a  $C \in \mathcal{S}(\mathcal{H})$  such that  $A \perp C$  and  $A \oplus C = B$ , namely,  $A + C = B$ .

Let  $x \in \mathcal{D}(A^2)$  which implies  $x \in \mathcal{D}(A)$  and  $Ax \in \mathcal{D}(A)$ . Since  $A \perp C$ ,  $\overline{\text{ran}}(A) \subseteq$

$null(C)$ . So  $Ax \in \mathcal{D}(C)$ . Then, we have  $Ax \in \mathcal{D}(A) \cap \mathcal{D}(C) = \mathcal{D}(B)$  and  $A^2x + CAx = BAx$ . As  $CAx = 0$ , it follows that  $A^2x = BAx$  for each  $x \in \mathcal{D}(A^2)$ . Thus  $A^2 \subseteq BA$ . To the contrary, suppose  $x \in \mathcal{D}(BA)$ . Then  $x \in \mathcal{D}(A)$  and  $Ax \in \mathcal{D}(B)$ . As  $A \perp C$ ,  $\bigcup_{n=1}^{\infty} F_n^A F_n^C(\mathcal{H})$  is the common core for  $A$  and  $C$  and so it is also the core for  $A + C = B$ . Since  $AF_n^A F_n^C Ax + CF_n^A F_n^C Ax = BF_n^A F_n^C Ax$  and  $BF_n^A F_n^C Ax \rightarrow BAx$ ,  $CF_n^A F_n^C Ax \rightarrow CAx = 0$ , we have  $AF_n^A F_n^C Ax \rightarrow BAx$ . As  $F_n^A F_n^C Ax \rightarrow Ax$  and  $A$  is closed, we obtain  $Ax \in \mathcal{D}(A)$  and  $A^2x = BAx$ . It follows that  $BA \subseteq A^2$ . Thus, we conclude  $A^2 = BA$ .

Sufficiency. Since  $A^2 = BA$ ,  $(B - A)A \subseteq 0$ . By Proposition 1,  $\mathcal{D}(B) \subseteq \mathcal{D}(A)$  and  $\mathcal{D}(B - A)$  is sense in  $\mathcal{H}$ . As  $B - A = B^* - A^* \subseteq (B - A)^*$ ,  $B - A$  is symmetric which implies that  $B - A$  is closable. Define  $C = \overline{B - A}$ . From Proposition 1,  $\bigcup_{n=1}^{\infty} F_n^B(\mathcal{H})$  is the common core for  $A$  and  $B$ . Thus it follows that  $\bigcup_{n=1}^{\infty} F_n^B(\mathcal{H})$  is the core for  $C$ . So  $F_n^B C \subseteq CF_n^B$ . Then  $C^* F_n^B \supseteq F_n^B C^*$ . That is  $\bigcup_{n=1}^{\infty} F_n^B(\mathcal{H})$  is the core for  $C^* = (\overline{B - A})^*$ . Since  $C = \overline{B - A} \subseteq (\overline{B - A})^*$ , we have  $C = C^*$  and  $C$  is self-adjoint. For  $x \in \mathcal{D}(A)$ ,  $F_n^A x \rightarrow x$  and  $AF_n^A x = F_n^A Ax \rightarrow Ax$ ,  $A^2 F_n^A x = BAF_n^A x$ . So  $CAF_n^A x = BAF_n^A x - A^2 F_n^A x = 0 \rightarrow 0$ . As  $C$  is closed,  $Ax \in \mathcal{D}(C)$  and  $CAx = 0$ . Thus  $\text{ran}(A) \subseteq \text{null}(C)$  and  $\overline{\text{ran}}(A) \subseteq \text{null}(C)$ . Then  $C \perp A$  and  $A \oplus C = A + C$ . As  $B \subseteq A + \overline{B - A} = A + C$  and  $A + C$  is self-adjoint. From the fact that a self-adjoint operator is maximal symmetric, we have  $B = A + \overline{B - A} = A + C$ . That is,  $A \preceq B$ .

From the above fact, we can conclude that  $A \preceq B$  implies  $\mathcal{D}(B) \subseteq \mathcal{D}(A)$ .

**Proposition 4.** For each  $A \in S(\mathcal{H})$ ,  $A$  is principal. That is, if  $B, C \in S(\mathcal{H})$  with  $B, C \preceq A$  and  $B \perp C$ , then  $B \oplus C \preceq A$ .

**Proof.** Let  $B, C \in S(\mathcal{H})$  with  $B, C \preceq A$  and  $B \perp C$ . Then by Proposition 3,  $B^2 = AB$  and  $C^2 = AC$ . From Proposition 1,  $\bigcup_{n=1}^{\infty} F_n^A(\mathcal{H})$  is the common core for  $A, B$  and  $C$ . So  $\mathcal{D}(A - (B + C))$  is a dense subspace of  $\mathcal{H}$ . As  $A - (B + C) \subseteq (A - (B + C))^*$ ,  $A - (B + C)$  is closable. Define  $H = \overline{A - (B + C)}$ . Just as the proof in Proposition 3, we can prove that  $H$  is self-adjoint. For each  $x \in \mathcal{D}(B + C)$ ,  $F_n^A x \rightarrow x$  and  $(B + C)F_n^A x = F_n^A(B + C)x \rightarrow (B + C)x$ . As  $B \perp C$ , it follows that

$H(B+C)F_n^A x = (A - (B+C))(B+C)F_n^A x = A(B+C)F_n^A x - (B^2 + C^2)F_n^A x = 0$ . As  $H$  is closed,  $(B+C)x \in \mathcal{D}(H)$  and  $H(B+C)x = 0$ . From Lemma 5,  $H \perp (B+C)$  and  $H \oplus (B+C) = \overline{A - (B+C)} + (B+C) = A$ . Hence,  $B \oplus C \preceq A$  and  $A$  is principle.

Recall that the traditional order of  $S(\mathcal{H})$ , we say  $A \leq B$  if  $\mathcal{D}(B) \subseteq \mathcal{D}(A)$  and  $\langle Ax, x \rangle \leq \langle Bx, x \rangle$  for each  $x \in \mathcal{D}(B)$ . We have the following result:

**Proposition 5.** If  $A \preceq B$  and  $B \geq 0$ , then  $A \leq B$ .

**Proof.** Suppose  $A \preceq B$ . Then  $\mathcal{D}(B) \subseteq \mathcal{D}(A)$  and there exists a  $C \in S(\mathcal{H})$  such that  $A \perp C$  and  $A \oplus C = B$ .  $\mathcal{H} = \overline{\text{ran}}(A) \oplus \text{null}(A)$ . For  $x \in \mathcal{D}(B)$ ,  $x = y + z$  where  $y \in \overline{\text{ran}}(A)$  and  $z \in \text{null}(A)$ . As  $x, z \in \mathcal{D}(A)$  implies  $y \in \mathcal{D}(A)$  and  $\overline{\text{ran}}(A) \subseteq \text{null}(C)$  implies  $y \in \mathcal{D}(C)$ . It follows that  $y \in \mathcal{D}(A) \cap \mathcal{D}(C)$ . Thus  $y \in \mathcal{D}(B)$  and  $z \in \mathcal{D}(B)$ . Then  $\langle (B-A)x, x \rangle = \langle (B-A)(y+z), y+z \rangle = \langle (B-A)z, y+z \rangle = \langle z, (B-A)(y+z) \rangle = \langle z, Bz \rangle \geq 0$ . So  $\langle Ax, x \rangle \leq \langle Bx, x \rangle$  for each  $x \in \mathcal{D}(B)$ . Hence,  $A \leq B$ .

Next, we characterize the order  $\preceq$  with the spectral measure of self-adjoint operators.

**Theorem 3.** Let  $A, B \in S(\mathcal{H})$ . Then  $A \preceq B$  if and only if  $\Delta E_{\lambda_j}^A \leq \Delta E_{\lambda_j}^B$  where  $\Delta E_{\lambda_j}^A = E_{\lambda_j}^A - E_{\lambda_{j-1}}^A$ ,  $0 \notin (\lambda_{j-1}, \lambda_j]$  ( $j = 1, 2, 3, \dots$ ) and  $\{E_{\lambda}^A\}$  is the resolution of identity for  $A$ .

**Proof.** Necessity.  $A \preceq B$  implies that there exists a  $C \in S(\mathcal{H})$  such that  $A \perp C$  and  $A + C = B$ . Then  $\bigcup_{n=1}^{\infty} F_n^A F_n^C(\mathcal{H})$  is the common core for  $A$  and  $C$ . So  $\bigcup_{n=1}^{\infty} F_n^A F_n^C(\mathcal{H})$  is also the core for  $A + C = B$ . By Proposition 3, we have  $A^2 = BA$ . While  $A^2 = BA$  implies  $(AF_n^A F_n^C)^2 = (BF_n^A F_n^C)(AF_n^A F_n^C)$  for each  $n \in \mathbb{N}$ . By Proposition 3 again, we have  $AF_n^A F_n^C \preceq BF_n^A F_n^C$  for each  $n \in \mathbb{N}$ . As  $\{E_{\lambda}^A F_n^A F_n^C\}$  and  $\{E_{\lambda}^B F_n^A F_n^C\}$  are the resolutions of identity for  $AF_n^A F_n^C|_{F_n^A F_n^C(\mathcal{H})}$  and  $BF_n^A F_n^C|_{F_n^A F_n^C(\mathcal{H})}$  respectively, by Theorem 4.6 in [10], we have  $\Delta E_{\lambda_j}^A F_n^A F_n^C \leq \Delta E_{\lambda_j}^B F_n^A F_n^C$  for each  $n \in \mathbb{N}$  and  $0 \notin (\lambda_{j-1}, \lambda_j]$ . Since  $F_n^A F_n^C \uparrow$  with the strong operator limit  $I$ , it follows that  $\Delta E_{\lambda_j}^A \leq \Delta E_{\lambda_j}^B$ ,  $0 \notin (\lambda_{j-1}, \lambda_j]$ .

Sufficiency. Suppose  $\Delta E_{\lambda_j}^A \leq \Delta E_{\lambda_j}^B$ ,  $0 \notin (\lambda_{j-1}, \lambda_j]$ . For each  $n \in \mathbb{N}$ ,  $\{E_{\lambda}^A F_n^A\}$

and  $\{E_\lambda^B F_n^B\}$  are the resolution of identity for  $AF_n^A|F_n^A(\mathcal{H})$  and  $BF_n^B|F_n^B(\mathcal{H})$  respectively. For each  $\Delta E_{\lambda_j}^A$  and  $\Delta E_{\lambda_j}^B$  with  $0 \notin (\lambda_{j-1}, \lambda_j]$ , as for each  $n \in \mathbf{N}$ , there exists  $0 \notin (\lambda_{j-1}, \lambda_j]$  such that  $\Delta E_{\lambda_j}^A \leq F_n^A$  and  $\Delta E_{\lambda_j}^B \leq F_n^B$ , it follows that  $\Delta E_{\lambda_j}^A F_n^A \leq \Delta E_{\lambda_j}^B F_n^B$ . Then  $AF_n^A \preceq BF_n^B$  and  $(AF_n^A)^2 = (BF_n^B)(AF_n^A)$ . For each  $x \in \mathcal{D}(A^2)$ ,  $x \in \mathcal{D}(A)$  and  $Ax \in \mathcal{D}(A)$ . As  $F_n^B F_n^A Ax \rightarrow Ax$ , we have  $BF_n^B F_n^A Ax = BF_n^B AF_n^A x = (AF_n^A)^2 x = F_n^A A^2 x \rightarrow A^2 x$ . Since  $B$  is closable,  $Ax \in \mathcal{D}(B)$  and  $BAx = A^2 x$ . So  $A^2 \subseteq BA$ . Conversely, for each  $x \in \mathcal{D}(BA)$ ,  $x \in \mathcal{D}(A)$  and  $Ax \in \mathcal{D}(B)$ . As  $(AF_n^A)^2$  is self-adjoint, we have  $(BF_n^B)(AF_n^A) = (AF_n^A)(BF_n^B)$ . By Lemma 4, we have  $BF_n^B F_n^A = F_n^A BF_n^B$ . Since  $F_n^A Ax \rightarrow Ax$ , we have  $AF_n^A Ax = (AF_n^A)^2 x = (BF_n^B)(AF_n^A)x = F_n^B BF_n^A Ax = F_n^B F_n^A BAx \rightarrow BAx$ . As  $A$  is closable,  $Ax \in \mathcal{D}(A)$  and  $A^2 x = BAx$ . So  $BA \subseteq A^2$ . Therefore,  $A^2 = BA$  which implies  $A \preceq B$ .

**Corollary 1.** Let  $A, B \in S(\mathcal{H})$ . Then  $A \preceq B$  if and only if  $P^A(\Delta) \leq P^B(\Delta)$  for each Borel set  $\Delta \subseteq \mathbf{R}$  with  $0 \notin \Delta$ .

Next, we study the exist of  $A \wedge B$  and  $A \vee B$  for  $A, B \in S(\mathcal{H})$ .

Denote  $\mathcal{B}(\mathbf{R})$  the set of all Borel subsets of  $\mathbf{R}$ . For each  $\Delta \in \mathcal{B}(\mathbf{R})$ , if  $\Delta = \cup_{i=1}^n \Delta_i$ , where  $\{\Delta_i\}_{i=1}^n$  are pairwise disjoint Borel subsets of  $\mathbf{R}$ , then we say  $\gamma = \{\Delta_i\}_{i=1}^n$  is a partition of  $\Delta$ . We denote all the partitions of  $\Delta$  by  $\Gamma(\Delta)$ .

Let  $A, B \in S(\mathcal{H})$ . Define  $P(\emptyset) = 0$ . For each nonempty  $\Delta \in \mathcal{B}(\mathbf{R})$  and  $\gamma \in \Gamma(\Delta)$ , define

$$P(\Delta) = \begin{cases} \bigwedge_{\gamma \in \Gamma(\Delta)} \sum_{\Delta_i \in \gamma} (P^A(\Delta_i) \wedge P^B(\Delta_i)) & 0 \notin \Delta \\ I - P(\mathbf{R} \setminus \Delta) & 0 \in \Delta \end{cases}.$$

**Lemma 8** ([15]).  $P : \mathcal{B}(\mathbf{R}) \rightarrow P(\mathcal{H})$  is a spectral measure.

**Theorem 4.** Let  $A, B \in S(\mathcal{H})$ . Then  $A \wedge B$  exists in  $S(\mathcal{H})$  with respect to  $\preceq$ .

**Proof.** Let  $\{E_\lambda^A\}, \{E_\lambda^B\}$  be the resolutions of identity for  $A$  and  $B$  respectively and  $P^A, P^B$  the spectral measures for  $A$  and  $B$  respectively. Define  $P(\Delta)$  as above for each Borel set  $\Delta \in \mathcal{B}(\mathbf{R})$  and then  $P$  is a spectral measure. Define  $E_\lambda = P((-\infty, \lambda])$  and  $\{E_\lambda\}$  is a resolution of identity. By Lemma 7, there exists a self-

adjoint  $C$  such that

$$Cx = \int_{-n}^n \lambda dE_\lambda x$$

and  $\{E_\lambda\}$  is the resolution of identity for  $C$ , where  $x \in F_n(\mathcal{H})$  and all  $n$ . Let  $\Delta \in \mathcal{B}(\mathbf{R})$  with  $0 \notin \Delta$ . For each  $\gamma \in \Gamma(\Delta)$ ,

$$\begin{aligned} P_\gamma &= \sum_{\Delta_i \in \gamma} (P^A(\Delta_i) \wedge P^B(\Delta_i)) \\ &\leq \left( \sum_{\Delta_i \in \gamma} (P^A(\Delta_i)) \right) \wedge \left( \sum_{\Delta_i \in \gamma} (P^B(\Delta_i)) \right) \\ &= P^A(\Delta) \wedge P^B(\Delta). \end{aligned}$$

Then  $P(\Delta) = \bigwedge_{\gamma \in \Gamma(\Delta)} P_\gamma \leq P^A(\Delta) \wedge P^B(\Delta)$ . From Corollary 1,  $C \preceq A$  and  $C \preceq B$ . Suppose there exists another  $C_1 \in S(\mathcal{H})$  such that  $C_1 \preceq A$  and  $C_1 \preceq B$ . For each  $\Delta \in \mathcal{B}(\mathbf{R})$  with  $0 \notin \Delta$  and  $\gamma \in \Gamma(\Delta)$ , since  $P^{C_1}(\Delta_i) \leq P^A(\Delta_i)$  and  $P^{C_1}(\Delta_i) \leq P^B(\Delta_i)$  for each Borel subsets  $\Delta_i \in \gamma$ , we have

$$P^{C_1}(\Delta) = \sum_{\Delta_i \in \gamma} P^{C_1}(\Delta_i) \leq \sum_{\Delta_i \in \gamma} P^A(\Delta_i) \wedge P^B(\Delta_i).$$

So we obtain

$$P^{C_1}(\Delta) \leq \bigwedge_{\gamma \in \Gamma(\Delta)} \sum_{\Delta_i \in \gamma} P^A(\Delta_i) \wedge P^B(\Delta_i) = P(\Delta).$$

Thus  $C_1 \preceq C$  and  $C = A \wedge B$ .

**Remark 1.** If  $\{A_\alpha\}_{\alpha \in \Lambda} \subseteq S(\mathcal{H})$ , then  $A = \bigwedge_{\alpha} A_\alpha$  exists in  $S(\mathcal{H})$ . In fact, define  $P(\emptyset) = 0$  and for each nonempty  $\Delta \in \mathcal{B}(\mathbf{R})$ ,

$$P(\Delta) = \begin{cases} \bigwedge_{\gamma \in \Gamma(\Delta)} \sum_{\Delta_i \in \gamma} \left( \bigwedge_{\alpha} P^{A_\alpha}(\Delta_i) \right) & 0 \notin \Delta \\ I - P(\mathbf{R} \setminus \Delta) & 0 \in \Delta \end{cases}.$$

It is clear that for each  $\Delta \in \mathcal{B}(\mathbf{R})$ ,  $P(\Delta)$  is a projection operator. It can be proved that  $P : \mathcal{B}(\mathbf{R}) \rightarrow P(\mathcal{H})$  is a spectral measure. Define  $E_\lambda = P((-\infty, \lambda])$  and  $\{E_\lambda\}$  is a resolution of identity. Then we have  $A = \bigwedge_{\alpha} A_\alpha$  where  $Ax = \int_{-n}^n \lambda dE_\lambda x$  for each  $x \in F_n(\mathcal{H})$  and all  $n$ .

Let  $A, B \in S(\mathcal{H})$ . Then  $A \preceq B$  implies  $P^A(\Delta) \leq P^B(\Delta)$  for each Borel subset  $\Delta$  with  $0 \notin \Delta$ . So  $P^A(\Delta) = P^A(\Delta)P^B(\Delta) = P^B(\Delta)P^A(\Delta)$ .  $P^A(\Delta_1)P^A(\Delta_2) = 0$

for the Borel subsets  $\Delta_1$  and  $\Delta_2$  with  $\Delta_1 \cap \Delta_2 = \emptyset$ . Therefore, the following result is straightforward.

**Lemma 9.** Let  $A, B \in S(\mathcal{H})$  and  $H \in S(\mathcal{H})$  be an upper bound of  $A$  and  $B$  with respect to  $\preceq$ . Then, for any two Borel subsets  $\Delta_1$  and  $\Delta_2$  of  $\mathbf{R}$ , if  $\Delta_1 \cap \Delta_2 = \emptyset$ ,  $0 \notin \Delta_1$  and  $0 \notin \Delta_2$ , we have

$$P^A(\Delta_1)P^B(\Delta_2) = P^A(\Delta_1)P^H(\Delta_1)P^H(\Delta_2)P^B(\Delta_2) = 0.$$

**Lemma 10** ([16]). Let  $A, B \in S(\mathcal{H})$  and have the following property: for each pair  $\Delta_1, \Delta_2 \in \mathcal{B}(\mathbf{R})$ , where  $\Delta_1 \cap \Delta_2 = \emptyset$  and  $0 \notin \Delta_1 \cup \Delta_2$ , we have  $P^A(\Delta_1)P^B(\Delta_2) = 0$ . Then, the following mapping  $P : \mathcal{B}(\mathbf{R}) \rightarrow P(\mathcal{H})$  define a spectral measure:

$$P(\Delta) = \begin{cases} P^A(\Delta) \vee P^B(\Delta) & 0 \notin \Delta \\ P^A(\Delta \setminus \{0\}) \vee P^B(\Delta \setminus \{0\}) + N_A \wedge N_B & 0 \in \Delta \end{cases}.$$

**Theorem 5.** Let  $A, B \in S(\mathcal{H})$ . If there exists an  $C \in S(\mathcal{H})$  such that  $A \preceq C$  and  $B \preceq C$ , then  $A \vee B$  exists in  $S(\mathcal{H})$  with respect to  $\preceq$ .

**Proof.** We define  $P$  the same as in Lemma 10. Then,  $P$  is a spectral measure and  $E_\lambda = P((-\infty, \lambda])$  is a resolution of identity. By Lemma 7, there exists a self-adjoint operator  $D$  such that

$$Dx = \int_{-n}^n \lambda dE_\lambda x$$

for each  $x$  in  $F_n(\mathcal{H})$  and all  $n$  and  $\{E_\lambda\}$  is the resolution of the identity for  $D$ , where  $F_n = E_n - E_{-n}$ . Clearly,  $P^A(\Delta) \leq P(\Delta)$  and  $P^B(\Delta) \leq P(\Delta)$  for each Borel subset  $\Delta$  with  $0 \notin \Delta$ . It follows from Corollary 1,  $A \preceq C$  and  $B \preceq C$ . If there exists another operator  $C_1 \in S(\mathcal{H})$  satisfying  $A \preceq C_1$  and  $B \preceq C_1$ . Then  $P^A(\Delta) \leq P^{C_1}(\Delta)$ ,  $P^B(\Delta) \leq P^{C_1}(\Delta)$  and  $P^A(\Delta) \vee P^B(\Delta) \leq P^{C_1}(\Delta)$  for each  $\Delta \in \mathcal{B}(\mathbf{R})$  with  $0 \notin \Delta$ . Then  $P^D(\Delta) \leq P^{C_1}(\Delta)$  for each  $\Delta \in \mathcal{B}(\mathbf{R})$  with  $0 \notin \Delta$ . Therefore, by Corollary 1,  $C \preceq C_1$ . Thus  $C = A \vee B$ .

**Remark 2.** Let  $\{A_\alpha\}_{\alpha \in \Lambda} \subseteq S(\mathcal{H})$  and  $A_\alpha \preceq H$  for each  $\alpha \in \Lambda$ . Then,  $C = \bigvee_\alpha A_\alpha$  exists in  $S(\mathcal{H})$ . In fact, define

$$P(\Delta) = \begin{cases} \bigvee_{\alpha} P^{A_{\alpha}}(\Delta) & 0 \notin \Delta \\ \bigvee_{\alpha} P^{A_{\alpha}}(\Delta \setminus \{0\}) + \bigwedge_{\alpha} N_{A_{\alpha}} & 0 \in \Delta \end{cases}.$$

It can be proved that  $P : \mathcal{B}(\mathcal{R}) \rightarrow \mathcal{P}(\mathcal{H})$  defines a spectral measure.  $E_{\lambda} = P((-\infty, \lambda])$  is a resolution of identity. There exists a self-adjoint operator  $C$  such that  $Cx = \int_{-n}^n \lambda dE_{\lambda}x$  for each  $x$  in  $F_n(\mathcal{H})$  and all  $n$ . Then  $\bigvee_{\alpha} A_{\alpha} = C$ .

**Theorem 6.** Let  $\mathcal{H} = L^2(-\infty, +\infty)$ , the position operator  $Q$  and momentum operator  $P$  satisfy the Heisenberg's commutation relation  $QP - PQ = -i\hbar I$ . Then  $Q \wedge P = 0$  with respect to the order  $\preceq$ .

**Proof.** In fact, let  $A \preceq P$  and  $A \preceq Q$  for  $A \in S(\mathcal{H})$ . By Proposition 3,  $A^2 = PA$ ,  $A^2 = QA$ . Thus,  $A^3 = PA^2 = PQA$ ,  $A^3 = QA^2 = QPA$ . So  $PQA = QPA$ . Applying Heisenberg's commutation relation  $QP - PQ = -i\hbar I$ , we have

$$QPA - PQA = (QP - PQ)A = -i\hbar IA.$$

Since  $\bigcup_{n=1}^{\infty} F_n^A(\mathcal{H}) \subseteq \mathcal{D}(A^3) = \mathcal{D}(PQA) = \mathcal{D}(QPA)$ ,  $QPAx - PQAx = (QP - PQ)Ax = -i\hbar IAx$  for each  $x \in \bigcup_{n=1}^{\infty} F_n^A(\mathcal{H})$ . So  $Ax = 0$  for each  $x \in \bigcup_{n=1}^{\infty} F_n^A(\mathcal{H})$ , which implies that  $A = 0$ . Therefore, we have  $Q \wedge P = 0$ .

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